

State-space geometry, non-extremal black holes and Kaluza-Klein monopoles

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We examine the statistical nature of the charged anticharged non-extremal black holes in string theory. From the perspective of the intrinsic Riemannian Geometry, the first principle of the statistical mechanics shows that the stability properties of general nonextremal nonlarge charged black brane solutions are divulged from the positivity of the corresponding principle minors of the space-state metric tensor. Under the addition of the Kaluza-Klein monopoles, a novel aspect of the Gaussian fluctuations demonstrates that the canonical fluctuations can be ascertained without any approximation. We offer the state-space geometric implication for the most general non-extremal black brane configurations in string theory.

Keywords: Intrinsic Geometry; String Theory; Physics of black holes; Classical black holes; Quantum aspects of black holes, evaporation, thermodynamics; Higher-dimensional black holes, black strings, and related objects; Statistical Fluctuation; Flow Instability.

PACS numbers: 2.40.-Ky; 11.25.-w; 04.70.-s; 04.70.Bw; 04.70.Dy; 04.50.Gh; 5.40.-a; 47.29.Ky

Generic higher charged non-extremal black branes in string theory [1, 2, 3, 4, 5, 6, 7, 8] and M -theory [9, 10, 11, 12] possess rich state-space geometric structures. Some examples of such state-space configurations involve statistical properties of the extremal and non-extremal black branes [13, 14, 15, 16, 17, 18, 19, 20]. In this paper, we focus our attention on the thermodynamic perspectives of the higher charged anticharged black brane configurations in string theory. We wish to explicate the nature of the state-space pair correlation functions and the associated stability properties of the higher charged black brane solutions containing an ensemble of branes and antibranes. In the past, there have been several notions analyzed in condensed matter physics [21, 22, 23, 24, 25]. Here, we shall consider eight charged anticharged string theory black brane configurations and analyze the state-space pair correlation functions and their relative scaling relations. Given the definite state-space description of consistent non-extremal black brane macroscopic solutions, we expose (i) for what conditions the considered black hole configuration is stable, (ii) how its state-space correlation functions scale in terms of the numbers of the branes and antibranes. In sequel, we enlist the complete set of non-trivial relative state-space correlation functions of the nonextremal nonlarge charged anticharged black brane configurations. See for an introduction references [14, 15, 16, 18, 19, 20]. A similar analysis remains valid for the black holes in general relativity [27,], attractor black holes [31, 32, 33, 34, 35, 36, 37, 38, 39] and Legendre transformed finite parameter chemical configurations [40, 41], quantum field theory and hot QCD backgrounds [42, 43] with finite chemical potentials and the strongly coupled quarkonium configurations [44, 45].

Before analyzing the state-space properties of the eight parameter black brane configuration, let us first provide a brief introduction to the thermodynamic geometry [21, 30, 13]. From the perspective of the intrinsic geometry, the state-space geometry is defined as the thermodynamic geometry with a set of non-equilibrium coordinates, viz. the number of the branes and antibranes. In this framework, the state-space metric tensor is defined as the negative Hessian matrix of the black hole entropy $S(\vec{x})$. In general, the components of the metric tensor are defined as

$$g_{ij} := -\frac{\partial^2 S(\vec{x})}{\partial x^j \partial x^i}. \quad (1)$$

In the above definition, a state-space covariant vector $\vec{x} \in M_n$ shall be understood as the collection of the charges and anticharges (n_i, m_i) of the considered black hole. We shall show that the state-space geometry thus defined takes an account of the local thermodynamic interactions and possible global vacuum phase transitions. To illustrate the consideration of state-space geometry, let us explicate the case of two parameter black brane

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configurations. To do so, let the two parameters be the charge n and the anticharge m , then the components of the Ruppeiner metric tensor are

$$g_{nn} = -\frac{\partial^2 S}{\partial n^2}, \quad g_{nm} = -\frac{\partial^2 S}{\partial n \partial m}, \quad g_{mm} = -\frac{\partial^2 S}{\partial m^2}. \quad (2)$$

In this se-up, the components of the state-space metric tensor are associated to the respective statistical pair correlation functions. It is worth mentioning that the co-ordinates on the state-space manifold are the parameters of the microscopic boundary conformal field theory which is dual the black hole space-time solution. This is because the underlying state-space metric tensor comprises of the Gaussian fluctuations of the entropy which is the function of the number of the branes and antibranes. For the chosen black hole configuration, the local stability of the underlying statistical system requires both principle minors to be positive. In this case, the diagonal components of the state-space metric tensor, viz., $\{g_{x_i x_i} \mid x_i = (n, m)\}$ signify the heat capacities of the system. This requires that the diagonal components of the state-space metric tensor

$$g_{x_i x_i} > 0, \quad i = n, m \quad (3)$$

be positive definite. In this investigation, we discuss the significance of the above observation for the eight parameter non-extremal black brane configurations in string theory. From the notion of the relative scaling property, we demonstrate the nature of the brane-brane pair correlations. From the perspective of the intrinsic Riemannian geometry, the stability properties of the eight parameter black branes are examined from the positivity of the principle minors of the space-state metric tensor. For the Gaussian fluctuations of the two charge equilibrium statistical configurations, the existence of a positive definite volume form on the state-space manifold $(M_2(R), g)$ imposes such a global stability condition. In particular, the above configuration leads to a stable statistical basis, if the determinant of the state-space metric tensor

$$\|g\| = S_{nn}S_{mm} - S_{nm}^2 \quad (4)$$

remains positive. For the two charge black brane configurations, the geometric quantities corresponding to the underlying state-space manifold elucidates typical features of the Gaussian fluctuations about an ensemble of equilibrium brane microstates. Subsequently, we can further calculate the Christoffel connection Γ_{ijk} , Riemann curvature tensor R_{ijkl} , Ricci tensor R_{ij} and Ricci scalar R for the intrinsic state-space manifold. From the above viewpoint, the intrinsic scalar curvature accompanies information of the global correlation volume of the underlying statistical systems. In the case of the two charge black hole configurations, we have the following scalar curvature

$$R = \frac{1}{2\|g\|^2} (S_{mm}S_{nnn}S_{nmm} + S_{nm}S_{nnm}S_{nmm} + S_{nn}S_{nnm}S_{mmm} - S_{nm}S_{nnn}S_{mmm} - S_{nn}S_{nmm}^2 - S_{mm}S_{nnm}^2). \quad (5)$$

In this picture, the zero scalar curvature signifies that the informations on the event horizon of the black hole fluctuate independently of each other, while a divergent scalar curvature indicates vacuum phase transitions. This leads to the fact that an ensemble of highly correlated pixels of information can have vacuum phase transitions on the event horizon of the black hole. Ruppeiner has further interpreted the assumption “that all the statistical degrees of freedom of a black hole live on the black hole event horizon” as indicating that the state-space scalar curvature signifies the average number of correlated Planck areas on the event horizon of the black hole [21]. From the perspective of Mathur’s fuzzball proposal [46], the present consideration takes an account of the fact that the area of the event horizon is an integral multiple of the Planck area [47].

Interestingly, notice further that the state-space scalar curvature explicates the nature of the long range global phase transitions. In this sense, an ensemble of microstates corresponds to the black hole states, which are statistically (i) interacting, if the underlying state-space configuration has a non-zero scalar curvature and (ii) non-interacting, if the scalar curvature vanishes identically. Incrementally, one may note that the state-space configuration of the eight charge-anticharge black hole is attractive or repulsive, and weakly interacting in general. Subsequently, we shall demonstrate that the above eight charge-anticharge black hole yield a stable statistical configuration, if at most three of the parameters, viz., the brane numbers $\{n_i\}$, and the antibrane numbers $\{m_i\}$ are allowed to fluctuate.

With this introduction of the two dimensional intrinsic state-space geometry, we shall proceed to systematically analyze the underlying statistical structures of the higher charged black hole configurations in string theory. Following the notions we have developed in the Refs. [14, 15, 16, 18, 17, 19, 20], the subsequent analysis is devoted to the state-space geometric implications of the eight charge non-extremal black branes. In the

present work, we examine the nature of the state-space of the non-extremal black hole under the contribution of finitely many non-trivially circularly fibered KK-monopoles. In this process, we shall also enlist the complete set of non-trivial relative state-space correlation functions of the configurations considered in [16, 18]. In the past, there have been calculations of the entropy of the extremal, near-extremal and general nonextremal solutions [48, 49] in string theory. Inductively, the most general charge anticharge nonextremal black hole has the following entropy

$$S(n_1, m_1, n_2, m_2, n_3, m_3, n_4, m_4) = 2\pi \prod_{i=1}^4 (\sqrt{n_i} + \sqrt{m_i}) \quad (6)$$

For the distinct $i, i, k \in \{1, 2, 3, 4\}$, we find that the components of the metric tensor are

$$\begin{aligned} g_{n_i n_i} &= \frac{\pi}{2n_i^{3/2}} \prod_{j \neq i} (\sqrt{n_j} + \sqrt{m_j}), \\ g_{n_i n_j} &= -\frac{\pi}{2(n_i n_j)^{1/2}} \prod_{k \neq i \neq j} (\sqrt{n_k} + \sqrt{m_k}), \\ g_{n_i m_i} &= 0, \\ g_{n_i m_j} &= -\frac{\pi}{2(n_i m_j)^{1/2}} \prod_{k \neq i \neq j} (\sqrt{n_k} + \sqrt{m_k}), \\ g_{m_i m_i} &= \frac{\pi}{2m_i^{3/2}} \prod_{j \neq i} (\sqrt{n_j} + \sqrt{m_j}), \\ g_{m_i m_j} &= -\frac{\pi}{2(m_i m_j)^{1/2}} \prod_{k \neq i \neq j} (\sqrt{n_k} + \sqrt{m_k}). \end{aligned} \quad (7)$$

From the above depiction, it is evident that the principle components of the state-space metric tensor $\{g_{n_i n_i}, g_{m_i m_i} \mid i = 1, 2, 3, 4\}$ essentially signify a set of definite heat capacities (or the related compressibilities) whose positivity in turn apprises that the black brane solutions comply with an underlying equilibrium statistical configuration. For an arbitrary number of the branes $\{n_i\}$ and antibranes $\{m_i\}$, we find that the associated state-space metric constraints as the diagonal pair correlation functions remain positive definite. In particular, $\forall i \in \{1, 2, 3, 4\}$, it is clear that we have the following positivity conditions

$$g_{n_i n_i} > 0 \mid n_i, m_i > 0, \quad g_{m_i m_i} > 0 \mid n_i, m_i > 0. \quad (8)$$

As observed in [16, 18], we find that the ratios of diagonal components vary inversely with a multiple of a well-defined factor in the underlying parameters, viz., the charges and anticharges, which change under the Gaussian fluctuations, whereas the ratios involving off diagonal components in effect uniquely inversely vary in the parameters of the chosen set A_i of equilibrium black brane configurations. This suggests that the diagonal components weaken in a relatively controlled fashion into an equilibrium, in contrast with the off diagonal components which vary over the domain of associated parameters defining the D_1 - D_5 - P - KK non-extremal non-large charge configurations. In short, we can easily substantiate for the distinct $x_i := (n_i, m_i) \mid i \in \{1, 2, 3, 4\}$ describing eight (anti)charge string theory black holes that the relative pair correlation functions have distinct types of relative correlation functions. Apart from the zeros, infinities and similar factorizations, we see that the non-trivial relative correlation functions satisfy the following scaling relations

$$\begin{aligned} \frac{g_{x_i x_i}}{g_{x_j x_j}} &= \left(\frac{x_j}{x_i}\right)^{3/2}, \\ \frac{g_{x_i x_j}}{g_{x_k x_l}} &= \left(\frac{x_i x_j}{x_k x_l}\right)^{-1/2} \left(\frac{\prod_{p \neq i \neq j} \sqrt{n_p} + \sqrt{m_p}}{\prod_{q \neq k \neq l} \sqrt{n_q} + \sqrt{m_q}}\right), \\ \frac{g_{x_i x_i}}{g_{x_i x_k}} &= -\sqrt{\left(\frac{x_k}{x_i}\right)} \left(\frac{\prod_{p \neq i} \sqrt{n_p} + \sqrt{m_p}}{\prod_{q \neq i \neq k} \sqrt{n_q} + \sqrt{m_q}}\right). \end{aligned} \quad (9)$$

As noticed in the Refs. [16, 18], it is not difficult to analyze the statistical stability properties of the eight charged string theory non-extremal black holes. In particular, one can easily determine the principle minors associated

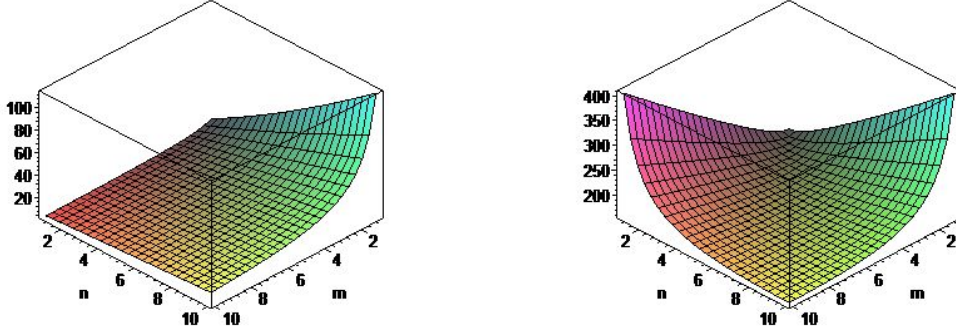


Figure 1: The heat capacities as the diagonal components g_{ii} of the state-space metric tensor and the surface minor p_2 , plotted as the functions of the number of branes n and antibranes m , describing the fluctuations in the statistical configuration.

with the state-space metric tensor and thereby argue that all the principle minors must be positive definite, in order to have a globally stable configuration. In the present case, it turns out that the above black hole is stable only when some of the charges and/or anticharges are held fixed or take specific values such that $p_i > 0$ for all the dimensions of the state-space manifold. Subsequently, from the definition of the Hessian matrix of the associated entropy concerning the most general nonextremal nonlarge charged black holes, we observe that some of the principle minors p_i are indeed non-positive. In fact, we discover an uniform local stability criteria on the lower dimensional hyper-surfaces and two dimensional surface of the underlying state-space manifold. The corresponding principle minors take the following explicit expressions

$$\begin{aligned}
p_1 &= \frac{\pi}{2n_1^{3/2}}(\sqrt{n_2} + \sqrt{m_2})(\sqrt{n_3} + \sqrt{m_3})(\sqrt{n_4} + \sqrt{m_4}), \\
p_2 &= \frac{\pi^2}{4(n_1 m_1)^{3/2}}(\sqrt{n_2} + \sqrt{m_2})^2(\sqrt{n_3} + \sqrt{m_3})^2(\sqrt{n_4} + \sqrt{m_4})^2, \\
p_3 &= \frac{\pi^3}{8(n_1 m_1 n_2)^{1/2}}(\sqrt{n_3} + \sqrt{m_3})^3(\sqrt{n_4} + \sqrt{m_4})^3(\sqrt{n_2} + \sqrt{m_2})(\sqrt{n_1} + \sqrt{m_1}), \\
p_4 &= 0, \\
p_5 &= -\frac{\pi^5}{8(n_1 n_2 m_2 m_1)^{3/2} n_3}(\sqrt{n_2} + \sqrt{m_2})(\sqrt{n_3} + \sqrt{m_3})^3(\sqrt{n_4} + \sqrt{m_4})^5(\sqrt{n_1} + \sqrt{m_1})^2 \tilde{p}_5, \\
p_6 &= -\frac{\pi^6}{16(n_1 n_2 m_1 m_2 n_3 m_3)^{3/2}}(\sqrt{n_2} + \sqrt{m_2})^2(\sqrt{n_3} + \sqrt{m_3})^3(\sqrt{n_4} + \sqrt{m_4})^6(\sqrt{n_1} + \sqrt{m_1})^3 \tilde{p}_6, \\
p_7 &= -\frac{\pi^7}{32(n_1 m_1 n_2 m_2 n_3 m_3 n_4)^{3/2}}(\sqrt{n_2} + \sqrt{m_2})^3(\sqrt{n_3} + \sqrt{m_3})^3(\sqrt{n_4} + \sqrt{m_4})^5(\sqrt{n_1} + \sqrt{m_1})^4 \tilde{p}_7, \\
p_8 &= -\frac{\pi^8}{16(\prod_{i=1}^4 n_i m_i)^{3/2}}(\sqrt{n_2} + \sqrt{m_2})^4(\sqrt{n_3} + \sqrt{m_3})^4(\sqrt{n_4} + \sqrt{m_4})^5(\sqrt{n_1} + \sqrt{m_1})^5 \tilde{p}_8. \tag{10}
\end{aligned}$$

The exact expression of the factors $\{\tilde{p}_i\}$ of the higher principle minors, viz., $\{p_i \mid i = 5, 6, 7, 8\}$ is relegated to the Appendix. As per the above evaluation, the graphical perspective of the state-space quantities is offered for the n branes and m antibranes. In fact, we have obtained the exact expressions for the components of the metric tensor, principle minors, determinant of the metric tensor and the underlying scalar curvature of the fluctuating statistical configuration of the eight parameter black holes in string theory. Qualitatively, the local and the global correlation properties of the limiting vacuum configuration are shown in Figs.(1, 2, 3, 4). Under the statistical fluctuations, the first three figures, viz., Figs.(1, 2, 3), describe the local stability properties, and the first of the last figure, viz., Fig.(4), describes the global ensemble stability, whereas the second one describes the corresponding phase space stability of the eight parameter black hole configuration.

In general, there exists an akin higher degree polynomial equation on which the Ricci scalar curvature becomes null, and exactly on these points the state-space configuration of the underlying non-large charge nonextremal eight charge black hole system corresponds to a non-interacting statistical system. Here, the state-space manifold (M_8, g) is curvature free. A systematic calculation further shows that the general expression for

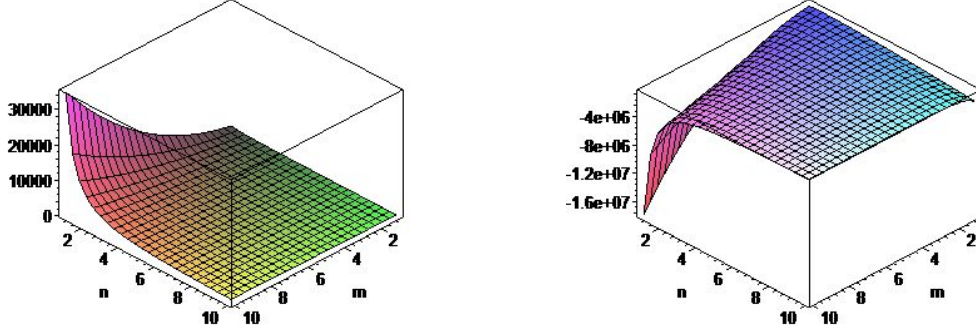


Figure 2: The hypersurface minors p_3 and p_5 of the state-space metric tensor, plotted as the functions of the number of branes n and antibranes m , describing the fluctuations in the statistical configuration.

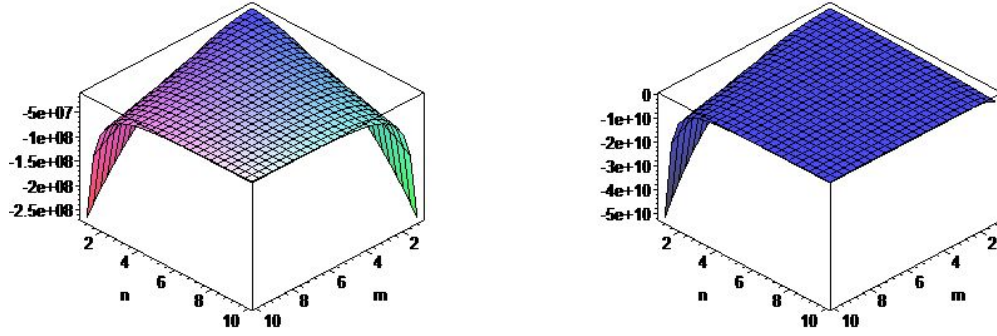


Figure 3: The hypersurface minors p_6 and p_7 of the state-space metric tensor, plotted as the functions of the number of branes n and antibranes m , describing the fluctuations in the statistical configuration.

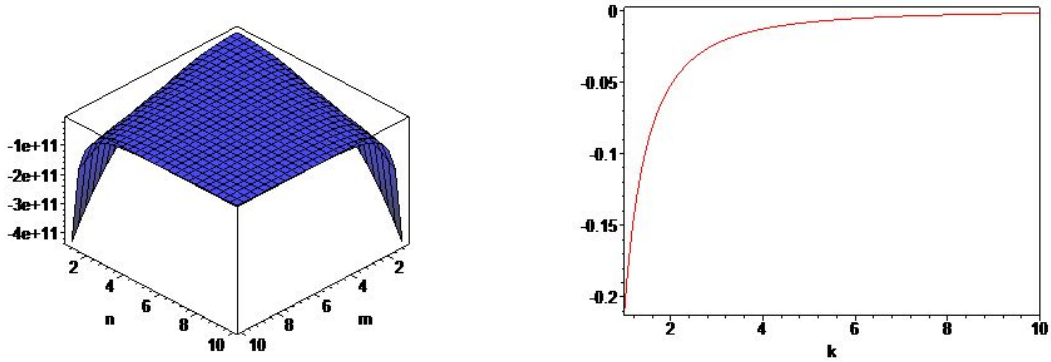


Figure 4: The determinant as the highest principle minor p_8 of the state-space metric tensor plotted as the function of the number of branes n and antibranes m , and the corresponding scalar curvature for the equal number of branes and antibranes ($n = m$) describing the fluctuations in the statistical configuration.

the Ricci scalar is quite involved, and even for equal brane charges $n_1 := n; n_2 := n; n_3 := n; n_4 := n$ and equal antibrane charges $m_1 := m; m_2 := m; m_3 := m, m_4 := m$ the result does not sufficiently simplify. Nevertheless, we find for the identical large values of brane and antibrane charges $n := k$ and $m := k$ [16, 18] that there exists an attractive state-space configuration. In the limit of a large k , the corresponding scalar curvature reduces to the following small negative value

$$R = -\frac{21}{32} \frac{1}{\pi k^2} \quad (11)$$

Interestingly, it turns out that the system becomes noninteracting in the limit of $k \rightarrow \infty$. For the case of the $n = k = m$, we observe that the corresponding principle minors reduce to the following constant values

$$\{p_i\}_{i=1}^8 = \{4\pi, 16\pi^2, 32\pi^3, 0, -2048\pi^5, -16384\pi^6, -163840\pi^7, -1048576\pi^8\}. \quad (12)$$

In this case, we find that the limiting underlying statistical system remains stable when at most three of the parameters, *viz.*, $\{n_i = n = m_i\}$, are allowed to fluctuate. Herewith, we find that the state-space manifold of the eight parameter brane and antibrane configuration is free from critical phenomena, except for the roots of the determinant. Thus, the regular state-space scalar curvature is comprehensively universal for the non-large charge non-extremal black brane configurations in string theory. In fact, the above perception turns out to be justified from the typical state-space geometry, *viz.*, the definition of the metric tensor as the negative Hessian matrix of the duality invariant expression of the black brane entropy. In this case, we may nevertheless easily observe, for a given entropy S_0 , that the constant entropy curve is given by the following curve

$$(\sqrt{n_1} + \sqrt{m_1})(\sqrt{n_2} + \sqrt{m_2})(\sqrt{n_3} + \sqrt{m_3})(\sqrt{n_4} + \sqrt{m_4}) = c. \quad (13)$$

where c is a real constant taking precisely the value $S_0/2\pi$. Under the vacuum fluctuations, the present analysis indicates that the entropy of the eight parameter black brane solution defines a non-degenerate embedding in the viewpoints of intrinsic state-space geometry. The above computations further encourage that our state-space geometry determines an intricate set of statistical properties, *viz.*, pair correlation functions and correlation volume, which reveal the possible nature of the associated parameters prescribing an ensemble of microstates of the dual CFT living on the boundary of the black brane solution. Furthermore, our expectation is that we can consider such an analysis for all higher dimensional black brane configurations with multiple parameters, where the state-space geometric propositions of having ordinary computations might be a bit involved. However, one may exhibit the intrinsic geometric acquisitions with an appropriate comprehension of the required parameters defining the state-space coordinate for the chosen black brane configurations in the string theory.

We have analyzed state-space pair correlation functions and the notion of stability for the non-extremal black holes in string theory. Our consideration is from the viewpoints of intrinsic state-space geometry. From the intrinsic Riemannian geometry, we find that the stability of these black branes has been divulged from the positivity of principle minors of the space-state metric tensor. Following developments introduced in the Refs. [14, 15, 16, 18, 17, 19, 20], we have explicitly extended the analysis of the state-space geometry for the four charge and four anticharge non-extremal black brane configurations in the string theory. The present consideration of the eight parameter black brane configurations, where the underlying leading order statistical entropy is written as a function of the charges $\{n_i\}$ and anticharges $\{m_i\}$ and describes the stability properties under the Gaussian fluctuations. The present consideration includes all the special cases of the extremal and near-extremal configurations with a fewer number of charges and anticharges. In this case, we obtain the standard pattern of the underlying state-space geometry and constant entropy curve as that of the lower parameter non-extremal black holes. In fact, the conclusion to be drawn remains the same, as the underlying state-space geometry remains well-defined as an intrinsic Riemannian manifold $N := M_8 \setminus \tilde{B}$, where \tilde{B} is the set of roots of the determinant of the metric tensor. The local coordinate of the state-space manifold involves the four charges and four anticharges of the underlying non-extremal black holes. Our analysis indicates that the leading order statistical behavior of the black brane configurations in string theory remains intact under the inclusion of the KK-monopoles.

Acknowledgement

This work has been supported in part by the European Research Council grant n. 226455, “*SUPERSYMMETRY, QUANTUM GRAVITY AND GAUGE FIELDS (SUPERFIELDS)*”.

This work was conducted during the period B.N.T. served as a postdoctoral research fellow at the “*INFN-Laboratori Nazionali di Frascati, Roma, Italy*”.

Appendix

In this appendix, we provide explicit forms of the higher principle minors of the state-space metric tensor of the eight charged nonextremal nonlarge black holes. Our analysis illustrates that the stability properties of the specific state-space hypersurface may exactly be exploited in general. The definite behavior of state-space properties, as accounted in the concerned main section suggests that the various intriguing hypersurfaces of the state-space configuration include the nice feature that they do have definite stability properties, except for some specific values of the charges and anticharges. As mentioned in the main sections, these configurations are generically well-defined and indicate an interacting statistical basis. Herewith, we discover that the state-space geometry of the general black brane configurations in string theory indicate the possible nature of the underlying statistical fluctuations. Significantly, we notice from the very definition of the intrinsic metric tensor that the related factors of the principle minors take the following expressions

$$\begin{aligned}
\tilde{p}_5 &:= n_2\sqrt{m_1} + 2\sqrt{n_2}\sqrt{m_2}\sqrt{m_1} + m_2\sqrt{m_1} + \sqrt{n_1}n_2 + 2\sqrt{n_1}\sqrt{n_2}\sqrt{m_2} + \sqrt{n_1}m_2, \\
\tilde{p}_6 &:= n_2\sqrt{m_1}\sqrt{n_3} + n_2\sqrt{m_1}\sqrt{m_3} + 2\sqrt{n_2}\sqrt{m_1}\sqrt{m_2}\sqrt{n_3} + 2\sqrt{n_2}\sqrt{m_1}\sqrt{m_2}\sqrt{m_3} \\
&\quad + \sqrt{m_1}m_2\sqrt{n_3} + \sqrt{m_1}m_2\sqrt{m_3} + \sqrt{n_1}n_2\sqrt{n_3} + \sqrt{n_1}n_2\sqrt{m_3} \\
&\quad + 2\sqrt{n_1}\sqrt{n_2}\sqrt{m_2}\sqrt{n_3} + 2\sqrt{n_1}\sqrt{n_2}\sqrt{m_2}\sqrt{m_3} + \sqrt{n_1}m_2\sqrt{n_3} + \sqrt{n_1}m_2\sqrt{m_3}, \\
\tilde{p}_7 &:= 2\sqrt{n_1}n_2\sqrt{n_3}\sqrt{m_3}\sqrt{m_4} + 2m_2\sqrt{n_3}\sqrt{m_3}\sqrt{m_4}\sqrt{m_1} + 8\sqrt{n_2}\sqrt{m_2}n_3\sqrt{n_4}\sqrt{m_1} \\
&\quad + 8\sqrt{n_1}\sqrt{n_2}\sqrt{m_2}n_3\sqrt{n_4} + 8m_2\sqrt{n_3}\sqrt{m_3}\sqrt{n_4}\sqrt{m_1} + 16\sqrt{n_2}\sqrt{m_2}\sqrt{n_3}\sqrt{m_3}\sqrt{n_4}\sqrt{m_1} \\
&\quad + 2\sqrt{n_2}\sqrt{m_2}n_3\sqrt{m_4}\sqrt{m_1} + 2\sqrt{n_1}\sqrt{n_2}\sqrt{m_2}m_3\sqrt{m_4} + 4\sqrt{n_1}\sqrt{n_2}\sqrt{m_2}\sqrt{n_3}\sqrt{m_3}\sqrt{m_4} \\
&\quad + 8n_2\sqrt{n_3}\sqrt{m_3}\sqrt{n_4}\sqrt{m_1} + 8\sqrt{n_1}n_2\sqrt{n_3}\sqrt{m_3}\sqrt{n_4} + 8\sqrt{n_1}m_2\sqrt{n_3}\sqrt{m_3}\sqrt{n_4} \\
&\quad + 8\sqrt{n_1}\sqrt{n_2}\sqrt{m_2}m_3\sqrt{n_4} + 8\sqrt{n_2}\sqrt{m_2}m_3\sqrt{n_4}\sqrt{m_1} + 2\sqrt{n_2}\sqrt{m_2}m_3\sqrt{m_4}\sqrt{m_1} \\
&\quad + 2\sqrt{n_1}\sqrt{n_2}\sqrt{m_2}n_3\sqrt{m_4} + 16\sqrt{n_1}\sqrt{n_2}\sqrt{m_2}\sqrt{n_3}\sqrt{m_3}\sqrt{n_4} + 2\sqrt{n_1}m_2\sqrt{n_3}\sqrt{m_3}\sqrt{m_4} \\
&\quad + \sqrt{n_1}n_2n_3\sqrt{m_4} + 4\sqrt{n_1}m_2m_3\sqrt{n_4} + n_2m_3\sqrt{m_4}\sqrt{m_1} + 4m_2n_3\sqrt{n_4}\sqrt{m_1} \\
&\quad + 4\sqrt{n_1}m_2n_3\sqrt{n_4} + \sqrt{n_1}m_2m_3\sqrt{m_4} + 4m_2m_3\sqrt{n_4}\sqrt{m_1} + 4n_2m_3\sqrt{n_4}\sqrt{m_1} \\
&\quad + m_2n_3\sqrt{m_4}\sqrt{m_1} + m_2m_3\sqrt{m_4}\sqrt{m_1} + \sqrt{n_1}n_2m_3\sqrt{m_4} + 4\sqrt{n_1}n_2m_3\sqrt{n_4} \\
&\quad + 4\sqrt{n_1}n_2n_3\sqrt{n_4} + \sqrt{n_1}m_2n_3\sqrt{m_4} + 4n_2n_3\sqrt{n_4}\sqrt{m_1} + n_2n_3\sqrt{m_4}\sqrt{m_1} \\
&\quad + 2n_2\sqrt{n_3}\sqrt{m_3}\sqrt{m_4}\sqrt{m_1} + 4\sqrt{n_2}\sqrt{m_2}\sqrt{n_3}\sqrt{m_3}\sqrt{m_4}\sqrt{m_1}, \\
\tilde{p}_8 &:= 2\sqrt{n_1}n_2\sqrt{n_3}\sqrt{m_3}\sqrt{m_4} + 2m_2\sqrt{n_3}\sqrt{m_3}\sqrt{m_4}\sqrt{m_1} + 2\sqrt{n_2}\sqrt{m_2}n_3\sqrt{n_4}\sqrt{m_1} \\
&\quad + 2\sqrt{n_1}\sqrt{n_2}\sqrt{m_2}n_3\sqrt{n_4} + 2m_2\sqrt{n_3}\sqrt{m_3}\sqrt{n_4}\sqrt{m_1} + 4\sqrt{n_2}\sqrt{m_2}\sqrt{n_3}\sqrt{m_3}\sqrt{n_4}\sqrt{m_1} \\
&\quad + 2\sqrt{n_2}\sqrt{m_2}n_3\sqrt{m_4}\sqrt{m_1} + 2\sqrt{n_1}\sqrt{n_2}\sqrt{m_2}m_3\sqrt{m_4} + 4\sqrt{n_1}\sqrt{n_2}\sqrt{m_2}\sqrt{n_3}\sqrt{m_3}\sqrt{m_4} \\
&\quad + 2n_2\sqrt{n_3}\sqrt{m_3}\sqrt{n_4}\sqrt{m_1} + 2\sqrt{n_1}n_2\sqrt{n_3}\sqrt{m_3}\sqrt{n_4} + 2\sqrt{n_1}m_2\sqrt{n_3}\sqrt{m_3}\sqrt{n_4} \\
&\quad + 2\sqrt{n_1}\sqrt{n_2}\sqrt{m_2}m_3\sqrt{n_4} + 2\sqrt{n_2}\sqrt{m_2}m_3\sqrt{n_4}\sqrt{m_1} + 2\sqrt{n_2}\sqrt{m_2}m_3\sqrt{m_4}\sqrt{m_1} \\
&\quad + 2\sqrt{n_1}\sqrt{n_2}\sqrt{m_2}n_3\sqrt{m_4} + 4\sqrt{n_1}\sqrt{n_2}\sqrt{m_2}\sqrt{n_3}\sqrt{m_3}\sqrt{n_4} + 2\sqrt{n_1}m_2\sqrt{n_3}\sqrt{m_3}\sqrt{m_4} \\
&\quad + \sqrt{n_1}n_2n_3\sqrt{m_4} + \sqrt{n_1}m_2m_3\sqrt{n_4} + n_2m_3\sqrt{m_4}\sqrt{m_1} + m_2n_3\sqrt{n_4}\sqrt{m_1} \\
&\quad + \sqrt{n_1}m_2n_3\sqrt{n_4} + \sqrt{n_1}m_2m_3\sqrt{m_4} + m_2m_3\sqrt{n_4}\sqrt{m_1} + n_2m_3\sqrt{n_4}\sqrt{m_1} \\
&\quad + m_2n_3\sqrt{m_4}\sqrt{m_1} + m_2m_3\sqrt{m_4}\sqrt{m_1} + \sqrt{n_1}n_2m_3\sqrt{m_4} + \sqrt{n_1}n_2m_3\sqrt{n_4} \\
&\quad + \sqrt{n_1}n_2n_3\sqrt{n_4} + \sqrt{n_1}m_2n_3\sqrt{m_4} + n_2n_3\sqrt{n_4}\sqrt{m_1} + n_2n_3\sqrt{m_4}\sqrt{m_1} \\
&\quad + 2n_2\sqrt{n_3}\sqrt{m_3}\sqrt{m_4}\sqrt{m_1} + 4\sqrt{n_2}\sqrt{m_2}\sqrt{n_3}\sqrt{m_3}\sqrt{m_4}\sqrt{m_1}.
\end{aligned} \tag{14}$$

References

- [1] A. Strominger, C. Vafa, “Microscopic Origin of the Bekenstein-Hawking Entropy”; Phys. Lett. B **379**, 99-104, (1996), [arXiv:hep-th/9601029v2](#).
- [2] A. Sen, “Black Hole Solutions in Heterotic String Theory on a Torus”; Nucl. Phys. B **440**, (1995) 421-440, [arXiv:hep-th/9411187v3](#).
- [3] A. Sen, “Extremal Black Holes and Elementary String States”; Mod. Phys. Lett. A **10**, (1995) 2081-2094, [arXiv:hep-th/9504147v2](#).

- [4] A. Dabholkar, “Exact Counting of Black Hole Microstates”; Phys. Rev. Lett. **94**, (2005) 241301, [arXiv:hep-th/0409148v2](#).
- [5] L. Andrianopoli, R. D’Auria, S. Ferrara, “Flat Symplectic Bundles of N-Extended Supergravities, Central Charges and Black-Hole Entropy”; [arXiv:hep-th/9707203v1](#).
- [6] A. Dabholkar, F. Denef, G. W. Moore, B. Pioline, “Precision Counting of Small Black Holes”; JHEP 0510 (2005) **096**, [arXiv:hep-th/0507014v1](#).
- [7] A. Dabholkar, F. Denef, G. W. Moore, B. Pioline, “Exact and Asymptotic Degeneracies of Small Black Holes”; JHEP 0508 (2005) **021**, [arXiv:hep-th/0502157v4](#).
- [8] A. Sen, “Stretching the Horizon of a Higher Dimensional Small Black Hole”; JHEP 0507 (2005) **073**, [arXiv:hep-th/0505122v2](#).
- [9] J. P. Gauntlett, Jan B. Gutowski, C. M. Hull, S. Pakis, H. S. Reall, “All supersymmetric solutions of minimal supergravity in five dimensions”; Class. Quant. Grav. **20**, (2003) 4587-4634, [arXiv:hep-th/0209114v3](#).
- [10] Jan B. Gutowski, H. S. Reall, “General supersymmetric AdS5 black holes”; JHEP 0404 (2004) **048**, [arXiv:hep-th/0401129v3](#).
- [11] I. Bena, N. P. Warner, “One Ring to Rule Them All ... and in the Darkness Bind Them?”; Adv. Theor. Math. Phys. **9** (2005) 667-701, [arXiv:hep-th/0408106v2](#).
- [12] J. P. Gauntlett, Jan B. Gutowski, “General Concentric Black Rings”; Phys. Rev. D **71** (2005) 045002, [arXiv:hep-th/0408122v3](#).
- [13] T. Sarkar, G. Sengupta, B. N. Tiwari, “On the Thermodynamic Geometry of BTZ Black Holes”; JHEP 0611 (2006) **015**, [arXiv:hep-th/0606084v1](#).
- [14] B. N. Tiwari, “Sur les corrections de la géométrie thermodynamique des trous noirs”, [arXiv:0801.4087v1 \[hep-th\]](#).
- [15] T. Sarkar, G. Sengupta, B. N. Tiwari, “Thermodynamic Geometry and Extremal Black Holes in String Theory”; JHEP 0810, **076**, 2008, [arXiv:0806.3513v1 \[hep-th\]](#).
- [16] S. Bellucci, B. N. Tiwari, “On the Microscopic Perspective of Black Branes Thermodynamic Geometry”; [arXiv:0808.3921v1 \[hep-th\]](#).
- [17] S. Bellucci, B. N. Tiwari, State-space correlations and stabilities, Phys. Rev. D **82**, 084008, (2010), [arXiv:0910.5309v1 \[hep-th\]](#).
- [18] S. Bellucci, B. N. Tiwari, “An exact fluctuating 1/2-BPS configuration”; JHEP **05** (2010) 023, [arXiv:0910.5314v2 \[hep-th\]](#).
- [19] S. Bellucci, B. N. Tiwari, “State-space Manifold and Rotating Black Holes”, [arXiv:1010.1427v1 \[hep-th\]](#).
- [20] S. Bellucci, B. N. Tiwari, “Black Strings, Black Rings and State-space Manifold”, [arXiv:10.3832v1 \[hep-th\]](#).
- [21] G. Ruppeiner, “Riemannian geometry in thermodynamic fluctuation theory”; Rev. Mod. Phys **67** (1995) 605, Erratum 68 (1996) 313.
- [22] G. Ruppeiner, “Thermodynamics: A Riemannian geometric model”; Phys. Rev. A **20**, 1608 (1979).
- [23] G. Ruppeiner, “Thermodynamic Critical Fluctuation Theory?”; Phys. Rev. Lett. **50**, 287 (1983).
- [24] G. Ruppeiner, “New thermodynamic fluctuation theory using path integrals”; Phys. Rev. A **27**, 1116, 1983.
- [25] G. Ruppeiner and C. Davis, “Thermodynamic curvature of the multicomponent ideal gas”; Phys. Rev. A **41**, 2200, 1990.

- [26] G. Ruppeiner, “Thermodynamic curvature and phase transitions in Kerr-Newman black holes”; *Phy. Rev. D* **78**, 024016 (2008)
- [27] J. E. Aman, I. Bengtsson, N. Pidokrajt, “Flat Information Geometries in Black Hole Thermodynamics”; *Gen. Rel. Grav.* **38** (2006) 1305-1315, [arXiv:gr-qc/0601119v1](#).
- [28] J. Shen, R. G. Cai, B. Wang, R. K. Su, “Thermodynamic Geometry and Critical Behavior of Black Holes”; *Int. J. Mod. Phys. A* **22** (2007) 11-27, [arXiv:gr-qc/0512035v1](#).
- [29] J. E. Aman, I. Bengtsson, N. Pidokrajt, “Geometry of black hole thermodynamics”; *Gen. Rel. Grav.* **35** (2003) 1733, [arXiv:gr-qc/0304015v1](#).
- [30] J. E. Aman, N. Pidokrajt, “Geometry of Higher-Dimensional Black Hole Thermodynamics”; *Phys. Rev. D* **73** (2006) 024017, [arXiv:hep-th/0510139v3](#).
- [31] S. Ferrara, R. Kallosh, A. Strominger, “N=2 Extremal Black Holes”; *Phys. Rev. D* **52** (1995) R5412-R5416, [arXiv:hep-th/9508072v3](#).
- [32] A. Strominger, “Macroscopic Entropy of $N = 2$ Extremal Black Holes”; *Phys. Lett. B* **383** (1996) 39-43, [arXiv:hep-th/9602111v3](#).
- [33] S. Ferrara, R. Kallosh, “N=2 Extremal Black Holes”; *Phys. Rev. D* **54** (1996) 1514-1524, [arXiv:hep-th/9602136](#).
- [34] S. Ferrara, G. W. Gibbons, R. Kallosh, “N=2 Extremal Black Holes”; *Nucl. Phys. B* **500** (1997) 75-93, [arXiv:hep-th/9702103](#).
- [35] S. Bellucci, S. Ferrara, A. Marrani, “Attractor Horizon Geometries of Extremal Black Holes”; Contribution to the Proceedings of the XVII SIGRAV Conference, 4-7 September 2006, Turin, Italy, [arXiv:hep-th/0702019](#).
- [36] S. Bellucci, S. Ferrara, A. Marrani, “Attractors in Black”; *Fortsch. Phys.* **56** (2008) 761, [arXiv:0805.1310](#).
- [37] S. Bellucci, S. Ferrara, M. Günaydin and A. Marrani, *SAM Lectures on Extremal Black Holes in $d = 4$ Extended Supergravity*, [arXiv:0905.3739 \[hep-th\]](#).
- [38] S. Bellucci, S. Ferrara, R. Kallosh and A. Marrani, *Extremal Black Hole and Flux Vacua Attractors*, *Lect. Notes Phys.* **755**, 115 (2008), [arXiv:0711.4547 \[hep-th\]](#).
- [39] S. Bellucci, S. Ferrara, A. Marrani, “Supersymmetric mechanics. Vol. 2: The attractor mechanism and space time singularities,” *Lect. Notes Phys.* **701** (2006) 1-225.
- [40] F. Weinhold, “Metric geometry of equilibrium thermodynamics”; *J. Chem. Phys.* **63** , 2479 (1975), DOI:10.1063/1.431689.
- [41] F. Weinhold, “Metric geometry of equilibrium thermodynamics. II: Scaling, homogeneity, and generalized Gibbs-Duhem relations”; *ibid J. Chem. Phys* **63** , 2484 (1975).
- [42] S. Bellucci, V. Chandra, B. N. Tiwari, “Thermodynamic Geometry and Free energy of Hot QCD”, To appear in *Int. J. Mod. Phys.*, [arXiv:0812.3792v1 \[hep-th\]](#).
- [43] S. Bellucci, V. Chandra, B. N. Tiwari, “A geometric approach to correlations and quark number susceptibilities”, [Contributed to the Conference on Quark confinement and the hadron spectrum QCHS IX, Madrid, Spain, 30 August 2010 - 03 September 2010. Proceedings will be published by the American Institute of Physics], [arXiv:1010.4405v1 \[hep-th\]](#).
- [44] S. Bellucci, V. Chandra, B. N. Tiwari, “Thermodynamic Stability of Quarkonium Bound States”, [arXiv:1010.4225v1 \[hep-th\]](#).
- [45] S. Bellucci, V. Chandra, B. N. Tiwari, “Strong Interactions, (De)coherence and Quarkonia”, [Contributed to the Conference on DISCRETE 2010, Rome, Italy, 06 December 2010 - 12 December 2010. Proceedings will be published by the Institute of Physics] [arXiv:1101.4745 \[hep-th\]](#).

- [46] S. D. Mathur, “The fuzzball proposal for black holes: an elementary review”; Fortsch. Phys. **53** (2005) 793-827, [arXiv:hep-th/0502050v1](#).
- [47] D. Bekenstein, “Information in the holographic universe”; Sci. Am. **289**, No. 2, 58-65 (2003).
- [48] G. T. Horowitz, D. A. Lowe, J. M. Maldacena: Statistical Entropy of Nonextremal Four-Dimensional Black Holes and U-Duality; Phys. Rev. Lett. 77 (1996) 430-433, [arXiv:hep-th/9603195v1](#).
- [49] C. V. Johnson, R. R. Khuri, R. C. Myers: Entropy of 4D Extremal Black Holes; Phys. Lett. B378 (1996) 78-86, [arXiv:hep-th/9603061v2](#).